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# MODELING OF PHASE TRANSITIONS OF THE FIRST KIND BY THE METHOD OF INTEGRAL EQUATIONS IN THE CASE OF A STATIONARY MOVING SURFACE SOURCE

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*The procedure for solving the Stefan problem by reducing it to a nonlinear integral equation of minimum dimensionality is described in brief. Examples of modeling the phase transitions are given when a half space and a metal layer are subjected to the thermal action of a stationary uniformly moving surface source of electric-arc origin.*

In practice, calculations in electrical engineering often include the problem of modeling the thermal action of electric-arc discharges on large conductors and shells. Analogous problems emerge in temperature calculations of materials heat-treated by lasers.

An adequate formulation of such problems leads to the well-known Stefan problem [1–4], while in the literature, calculation procedures are usually used that disregard the latent heat of melting (crystallization) [5, 6]. Compared to ordinary boundary-value problems (BVP) for a parabola-law second-order equation (the heat conduction equation) the Stefan problem is distinguished by the so-called Stefan nonlinearity of the equation with a singilarity of the delta-function type. The first-order derivative of a solution of the Stefan problem is a discontinuous function.

The enumerated circumstances cause additional difficulties in solving the Stefan problem by finite-difference and finite-element methods, which bring about certain requirements for smoothness of the equation and its solution [3, 4]. In the case of a multidimensional formulation of a BVP, the above methods need great expenditures of the working memory. The method of boundary-value integral equations is free of some of these drawbacks, which explains the keen interest of researchers to reducing the Stefan problem an equivalent integral equation of minimum dimensionality [1–4].

The algorithm for reduction and the computational scheme for solving the Stefan problem by the method of integral equations for a nonstationary multidimensional case is described in detail in [1]. The present work is devoted to the practically important case of a stationary uniformly moving surface source.

**1. Thermal Action of a Surface Source upon a Bulky Electrode.** We now consider the temperature condition for an electric arc whose reference spot moves uniformly and rectilinearily along the $x$ coordinate so that the coordinate points fixing the arc to the electrode surface are of the form $(x-vt, y, z = 0)$, where $v$ is the arc velocity, and $(x, y, z)$ are Cartesian coordinates.

If we introduce local coordinates associated with the center of the reference spot, then the boundary-value problem may be formulated as [1]:

$$- \operatorname{div} (\lambda \operatorname{grad} u) - \rho c v \frac{\partial u}{\partial x} = f + \rho \alpha v \frac{\partial \eta}{\partial x} (u - u_f) , \quad M \in R^3_- ,$$

$$\lambda \frac{\partial u}{\partial z} = q_S , \quad z = 0 ; \quad \lambda , \ \rho , \ c , \ \alpha , \ v = \text{const} ,$$

where $R^3_- = \{(x, y z) \mid z < 0\}$.

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The well-known procedure yields the following expression for the temperature distribution:

$$u(M) = H(M) - \rho a v \int\limits_{R_-^3} \frac{\partial G}{\partial x_N}(M, N)\, \eta\, (u(N) - u_f)\, dV_N, \quad M \in R_-^3.$$

Here $H(M)$ is a function describing the temperature field in the absence of the latent heat of melting; $G(M, N)$ is the Green function for a half-space:

$$G(M, N) = \exp\left[b(x_M - x_N)\right]\left\{\exp(-br_{MN})/r_{MN} + \exp(-br^*_{MN})/r^*_{MN}\right\}/(4\pi\lambda),$$

$$r^*_{MN} = \sqrt{(x_M - x_N)^2 + (y_M - y_N)^2 + (z_M + z_N)^2},$$

where $b$ is a parameter equal to $\rho c v/2\lambda$.

If we employ the a priori unknown function $z = z_0(x, y)$ describing, in Cartesian coordinates, the $z$ coordinate of the phase-transition surface, then the integral representation of temperature $u(x, y, z)$ acquires the form

$$u(x, y, z) = H(x, y, z) - \rho a v \int\limits_{R^2} \int\limits_{z_0(x_N, y_N)}^{0} \frac{\partial G}{\partial x_N}(x - x_N, y - y_N, z - z_N)\, dV_N.$$

By virtue of the equality

$$\frac{\partial}{\partial x_N} \int\limits_{z_0(x_N, y_N)}^{0} G(x - x_N, y - y_N, z - z_N)\, dz_N =$$

$$= \int\limits_{z_0(x_N, y_N)}^{0} \frac{\partial G}{\partial x_N}(x - x_N, y - y_N, z - z_N)\, dz_N -$$

$$- \frac{\partial z_0}{\partial x_N} G(x - x_N, y - y_N, z - z_0(x_N, y_N))$$

the three-dimensional integral in the last expression for $u(x, y, z)$ is transformed into a two-dimensional one:

$$u(x, y, z) = H(x, y, z) - \rho a v \int\limits_{R^2} G(x - x_N, y - y_N, z - z_0(x_N, y_N)) \frac{\partial z_0}{\partial x_N}\, dx_N\, dy_N.$$

$$u_f = H(x, y, z_0(x, y)) - \rho a v \int\limits_{R^2} G(x - x_N, y - y_N, z_0(x, y) - z_0(x_N, y_N)) \times$$

Assuming $z = z_0(x, y)$, we obtain the sought nonlinear integral equation relative to $z_0(x, y)$:

$$\times \frac{\partial z_0}{\partial x_N}\, dx_N\, dy_N, \quad (x, y) \in R^2.$$

It is sound practice to solve this equation by the quadrature method, but preliminarily it is necessary to pass to spherical coordinates.

**2. Thermal Action of a Surface Source upon a Thin Metal Layer.** The corresponding expression for the temperature distribution is as follows:

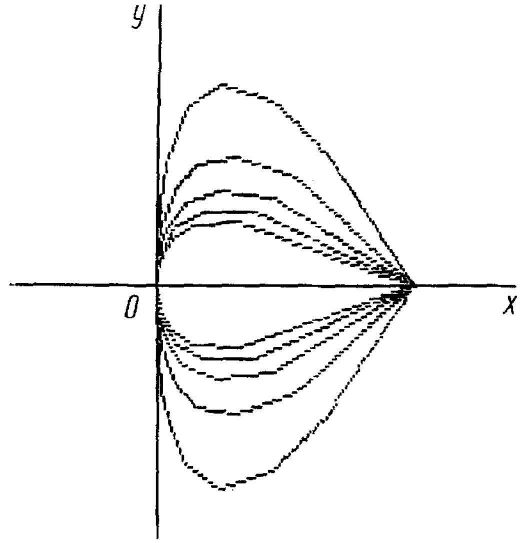


Fig. 1. Isotherms of layer melting under different heating conditions.

$$u(x, y) = H(x, y) - \rho\alpha v \int_{-\infty}^{\infty} dx_N \int_{-y_0(x_N)}^{y_0(x_N)} \frac{\partial G}{\partial x_N}(x - x_N, y - y_N)\, dy_N, \quad (x, y) \in R^2,$$

where $y_0(x) = r_0(\varphi)\sin\varphi$, $0 \le \varphi < 2\pi$; $G(M, N) = \exp[b(x_M - x_N)]\,K_0(br_{MN})$; $K_0(x)$ is the McDonald function.

Taking into account the equality

$$\frac{\partial}{\partial x_N} \int_{-y_0(x_N)}^{y_0(x_N)} G(x - x_N, y - y_N)\, dy_N = [G(x - x_N, y - y_0(x_N)) +$$

$$+ G(x - x_N, y + y_0(x_N))]\, y_0'(x_N) + \int_{-y_0(x_N)}^{y_0(x_N)} \frac{\partial G}{\partial x_N}(x_M - x_N, y_M - y_N)\, dy_N,$$

we obtain a representation of $u(x, y)$ in terms of a one-dimensional integral:

$$u(x, y) = H(x, y) + \rho\alpha v \int_{-\infty}^{\infty} [G(x - x_N, y - y_0(x_N)) +$$

$$+ G(x - x_N, y + y_0(x_N))]\, y_0'(x_N)\, dx_N,$$

whence the one-dimensional nonlinear integral equation for $y_0(x)$ or $r_0(\varphi)$ follows:

$$u_f = \frac{P}{2\pi\lambda h} K_0(br_0)\exp(bx) + \rho\alpha v \int_{-r_0(\pi)}^{r_0(0)} [G(x - x_N, y_0(x) - y_0(x_N)) +$$

$$+ G(x - x_N, y_0(x) + y_0(x_N))]\, y_0'(x_N)\, dx_N, \quad -r_0(\pi) \le x \le r_0(0),$$

where the expression for $H(x, y)$ corresponds to the case of a moving point source.

For numerical integration, we transform the integral equation by the quadrature method using the formula

$$u_f = \frac{P}{2\pi\lambda h} K_0(br_k)\exp(bx_k) + \rho\alpha v \sum_{j=1}^{m-1} [G(x_k - x_{j+1/2}, y_k - y_{j+1/2}) +$$

$$+ G(x_k - x_{j+1/2},\ y_k + y_{j+1/2})\ ](y_{j+1} - y_j), \quad k = 1, \ldots, m;$$

$$x_{j+1/2} = (x_j + x_{j+1})/2, \quad y_{j+1/2} = (y_j + y_{j+1})/2, \quad x_j = r_j \cos \varphi_j,$$

$$y_j = r_j \sin \varphi_j, \quad r_j \approx r_0(\varphi_j), \quad \varphi_j = \pi(m - j)/(m - 1), \quad j = 1, \ldots, m.$$

The unknown values of $r_1, \ldots, r_m$ are calculated according to the iteration scheme of the implicit Seidel method:

$$F_k(r_1^{(n+1)}, \ldots, r_{k-1}^{(n+1)}, r_k^{(n+1)}, r_{k+1}^{(n)}, \ldots, r_m^{(n)}) = 0, \quad k = 1, \ldots, m.$$

The results of the solution are represented by a series of computer graphs. In Fig. 1, each isotherm is drawn on a relative scale in which the semimajor axis of the melting isotherm is assumed to be unity. With increase in the velocity of displacement of the arc reference spot, the isotherms become more elongated. An analogous effect occurs when the thickness of the metal sheet decreases or the arc power increases.

Thus, the nature of the main dependences corresponds, qualitatively, to the well-known data in the theory of welding and electro-erosion processes. However, quantitatively, according to the present author's data, account for the latent heat of melting is necessary in many instances since it influences substantially the absolute geometric dimensions of the melting zone (for the majority of metals within 20–50%).

The method described is especially effective when realized with the aid of microprocessors and personal computers.

## NOTATION

$x$, abscissa; $y$, ordinate; $z$, third coordinate; $t$, time; $v$, velocity; $u$, absolute temperature; $\lambda$, thermal conductivity; $\rho$, mass density; $c$, heat capacity; $\alpha$, latent heat of the phase transition; $u_j$, temperature of the phase transition; $\eta$, Heaviside unit function; $M$, point of the three-dimensional space; $q_s$, density of the surface source; $f$, density of the bulk source; $r_{M,N}$, distance between the points $M$ and $N$; $\varphi$, polar angle; $P$, arc power; $h$, thickness of the metal layer; $m$, number of nodes of the finite-difference scheme; $n$, number of the iteration.

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